## Cramer's Rule for Solving Simultaneous Linear Equations

## Introduction

The need to solve systems of linear equations arises frequently in engineering. The analysis of electric circuits and the control of systems are two examples.
Cramer's rule for solving such systems involves the calculation of determinants and their ratio. For systems containing only a few equations it is a useful method of solution.

## Prerequisites

Before starting this Block you should ...
(1) be able to evaluate $2 \times 2$ and $3 \times 3$ determinants

## Learning Outcomes

After completing this Block you should be able to ...
$\checkmark$ State and apply Cramer's rule to find the solution of two simultaneous linear equations
$\checkmark$ State and apply Cramer's rule to find the solution of three simultaneous linear equations
$\checkmark$ Recognise cases where the solution is not unique or does not exist

## Learning Style

To achieve what is expected of you ...
allocate sufficient study time
briefly revise the prerequisite material
attempt every guided exercise and most of the other exercises

## 1. Solving two equations in two unknowns

If we have one linear equation

$$
a x=b
$$

in which the unknown is $x$ and $a$ and $b$ are constants then there are just three possibilities

- $a \neq 0$ then $x=\frac{b}{a} \equiv a^{-1} b$. The equation $a x=b$ has a unique solution for $x$.
- $a=0, b=0$ then the equation $a x=b$ becomes $0=0$ and any value of $x$ will do. There are infinitely many solutions to the equation $a x=b$.
- $a=0$ and $b \neq 0$ then $a x=b$ becomes $0=b$ which is a contradiction. In this case the equation $a x=b$ has no solution for $x$.

What happens if we have more than one equation and more than one unknown? We shall find that the solutions to such systems can be characterised in a manner similar to that occurring for a single equation; that is, a system may have a unique solution, an infinity of solutions or no solution at all.
In this block we examine a method, known as Cramer's rule and employing determinants, for solving systems of linear equations.
Consider the equations

$$
\begin{align*}
& a x+b y=e  \tag{i}\\
& c x+d y=f \tag{ii}
\end{align*}
$$

where $a, b, c, d, e, f$ are given numbers. The variables $x$ and $y$ are unknowns we wish to find. The values of $x$ and $y$ which simultaneously satisfy both equations are called solutions. Simple algebra will eliminate the variable $y$ between these equations. We multiply equation (i) by $d$, equation (ii) by $b$ and subtract:

$$
\begin{array}{ll}
\text { first, } & a d x+b d y=e d \\
\text { and } & b c x+b d y=b f
\end{array}
$$

(we multiplied in this way to identify the coefficients of $y$ as clearly equal.)
Now subtract to obtain

$$
\begin{equation*}
(a d-b c) x=e d-b f \tag{iii}
\end{equation*}
$$

Now do this exercise
Starting with equations (i) and (ii) eliminate $x$.

If we multiply this last equation by -1 we obtain

$$
\begin{equation*}
(a d-b c) y=a f-e c \tag{iv}
\end{equation*}
$$

Dividing equations (iii) and (iv) by $a d-b c$ we obtain the solutions

$$
\begin{equation*}
x=\frac{e d-b f}{a d-b c}, \quad y=\frac{a f-e c}{a d-b c} \tag{v}
\end{equation*}
$$

There is of course one proviso. If $a d-b c=0$ then neither $x$ nor $y$ has a defined value.
If we choose to express these solutions in terms of determinants we have the formulation for the solution of simultaneous equations known as Cramer's rule.

If we define $\Delta$ as the determinant $\left|\begin{array}{ll}a & b \\ c & d\end{array}\right|$ and provided $\Delta \neq 0$ then the unique solution of the equations

$$
\begin{aligned}
& a x+b y=e \\
& c x+d y=f
\end{aligned}
$$

is by (v) given by

$$
x=\frac{\Delta_{x}}{\Delta}, \quad y=\frac{\Delta_{y}}{\Delta} \quad \text { where } \quad \Delta_{x}=\left|\begin{array}{ll}
e & b \\
f & d
\end{array}\right|, \quad \Delta_{y}=\left|\begin{array}{ll}
a & e \\
c & f
\end{array}\right|
$$

Now $\Delta$ is the determinant of coefficients on the left-hand sides of the equations. In the expression $\Delta_{x}$ the coefficients of $x$ (i.e. $\binom{a}{c}$ which is column 1 of $\Delta$ ) are replaced by the terms on the right-hand sides of the equations (i.e. by $\binom{e}{f}$ ). Similarly in $\Delta_{y}$ the coefficients of $y$ (column 2 of $\Delta$ ) are replaced by the terms on the right-hand sides of the equation.

## Key Point

Cramer's Rule
The unique solution to the equations:

$$
\begin{aligned}
& a x+b y=e \\
& c x+d y=f
\end{aligned}
$$

is given by:

$$
x=\frac{\Delta_{x}}{\Delta}, \quad y=\frac{\Delta_{y}}{\Delta}
$$

in which

$$
\Delta=\left|\begin{array}{ll}
a & b \\
c & d
\end{array}\right| \quad \Delta_{x}=\left|\begin{array}{ll}
e & b \\
f & d
\end{array}\right|, \quad \Delta_{y}=\left|\begin{array}{ll}
a & e \\
c & f
\end{array}\right|
$$

If $\Delta=0$ this method of obtaining the solution cannot be used.

Now do this exercise
Use Cramer's rule to solve the simultaneous equations

$$
\begin{array}{r}
2 x+y=7 \\
3 x-4 y=5
\end{array}
$$

Now do this exercise
Repeat the process with the equations
(a)

$$
\begin{aligned}
& 2 x-3 y=6 \\
& 4 x-6 y=12
\end{aligned}
$$

(b) $\begin{aligned} 2 x-3 y & =6 \\ 4 x-6 y & =10\end{aligned}$

Notation For ease of generalisation to larger systems we write the two-equation system in a different notation:

$$
\begin{aligned}
& a_{11} x_{1}+a_{12} x_{2}=b_{1} \\
& a_{21} x_{2}+a_{22} x_{2}=b_{2}
\end{aligned}
$$

Here the unknowns are $x_{1}$ and $x_{2}$, the right-hand sides are $b_{1}$ and $b_{2}$ and the coefficients are $a_{i j}$ where, for example, $a_{21}$ is the coefficient of $x_{1}$ in equation two. In general, $a_{i j}$ is the coefficient of $x_{j}$ in equation $i$.
Cramer's rule can then be stated as follows:
If $\left|\begin{array}{ll}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right| \neq 0$, then the equations

$$
\begin{aligned}
a_{11} x_{1}+a_{12} x_{2} & =b_{1} \\
a_{21} x_{1}+a_{22} x_{2} & =b_{2}
\end{aligned}
$$

have solutions

$$
x_{1}=\frac{\left|\begin{array}{ll}
b_{1} & a_{12} \\
b_{2} & a_{22}
\end{array}\right|}{\left|\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right|}, \quad x_{2}=\frac{\left|\begin{array}{ll}
a_{11} & b_{1} \\
a_{21} & b_{2}
\end{array}\right|}{\left|\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right|} .
$$

## 2. Solving three equations in three unknowns

Cramer's rule can be extended to larger systems of simultaneous equations but the calculational effort increases rapidly as the size of the system increases.
We quote Cramer's rule for a system of three equations.

## Key Point

The unique solution to the system of equations:

$$
\begin{aligned}
& a_{11} x_{1}+a_{12} x_{2}+a_{13} x_{3}=b_{1} \\
& a_{21} x_{1}+a_{22} x_{2}+a_{23} x_{3}=b_{2} \\
& a_{31} x_{1}+a_{32} x_{2}+a_{33} x_{3}=b_{3}
\end{aligned}
$$

is

$$
x_{1}=\frac{\Delta_{x_{1}}}{\Delta}, \quad x_{2}=\frac{\Delta_{x_{2}}}{\Delta}, \quad x_{3}=\frac{\Delta_{x_{3}}}{\Delta}
$$

in which

$$
\Delta=\left|\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right|
$$

and

$$
\Delta_{x_{1}}=\left|\begin{array}{lll}
b_{1} & a_{12} & a_{13} \\
b_{2} & a_{22} & a_{23} \\
b_{3} & a_{32} & a_{33}
\end{array}\right| \quad \Delta_{x_{2}}=\left|\begin{array}{ccc}
a_{11} & b_{1} & a_{13} \\
a_{21} & b_{2} & a_{23} \\
a_{31} & b_{3} & a_{33}
\end{array}\right| \quad \Delta_{x_{3}}=\left|\begin{array}{ccc}
a_{11} & a_{12} & b_{1} \\
a_{21} & a_{22} & b_{2} \\
a_{31} & a_{32} & b_{3}
\end{array}\right|
$$

If $\Delta=0$ this method of obtaining the solution cannot be used.

Notice that the structure of the fractions is similar to that for the two-equation case. For example, the determinant forming the numerator of $x_{1}$ is obtained from the determinant of coefficients, $\Delta$, by replacing the first column by the right-hand sides of the equations.
Notice too the increase in calculation: in the two-equation case we had to evaluate three $2 \times 2$ determinants, whereas in the three-equation case we have to evaluate four $3 \times 3$ determinants. Hence Cramer's rule is not really practicable for larger systems.

Try each part of this exercise
We wish to solve the system

$$
\begin{aligned}
x_{1}-2 x_{2}+x_{3} & =3 \\
2 x_{1}+x_{2}-x_{3} & =5 \\
3 x_{1}-x_{2}+2 x_{3} & =12 .
\end{aligned}
$$

Part (a) First check that $\Delta \neq 0$.

Part (b) Now we find the value of $x_{1}$. First write down the expression for $x_{1}$ in terms of determinants.

Part (c) Now calculate $x_{1}$ explicitly.
Answer
Part (d) In a similar way find the values of $x_{2}$ and $x_{3}$.
Answer

More exercises for you to try

1. Solve the following using Cramer's rule:
(a) $\quad \begin{aligned} & 2 x-3 y=1 \\ & 4 x+4 y=2\end{aligned}$
(b) $\begin{aligned} 2 x-5 y & =2 \\ -4 x+10 y & =1\end{aligned}$
(c) $\begin{aligned} & 6 x-y=0 \\ & 2 x-4 y=1\end{aligned}$
2. Using Cramer's rule obtain the solutions to the following sets of equations:
(a) $\begin{aligned} 2 x_{1}+x_{2}-x_{3} & =0 \\ x_{1}+x_{3} & =4 \\ x_{1}+x_{2}+x_{3} & =0\end{aligned}$
(b) $\begin{aligned} & x_{1}-x_{2}+x_{3}=1 \\ &-x_{1}+x_{3}=1 \\ & x_{1}+x_{2}-x_{3}=0\end{aligned}$

## 3. Computer Exercise or Activity



For this exercise it will be necessary for you to access the computer package DERIVE.

Derive can be used to evaluate determinants, and so, indirectly, can be used to solve systems of linear equations using Cramer's rule.
To evaluate a determinant it must first be input in matrix form. For example to evaluate:

$$
\Delta=\left|\begin{array}{rr}
2 & 1 \\
3 & -4
\end{array}\right|
$$

we would key $\underline{\text { Author: }}$ Matrix. Then choose 2 rows and 2 columns and click OK. Now input the numbers as the screen directs and hit OK. DERIVE responds with
$\left|\begin{array}{rr}2 & 1 \\ 3 & -4\end{array}\right|$

Now copy this expression (using ctrl+c keys) and transfer this into the Author Expression dialog box. DERIVE responds with

$$
[[2,1],[3,-4]]
$$

Now put the cursor at the beginning of this line and type det to give the expression $\operatorname{det}[[2,1],[3,-4]]$. Hit the OK button and DERIVE responds
$\operatorname{DET}\left|\begin{array}{rr}2 & 1 \\ 3 & -4\end{array}\right|$
Now hit $\underline{\text { SimplifyBasic and DERIVE responds: }}$
-11
as we expect.

End of Block 8.1

Multiply equation (i) by $c$ and equation (ii) by $a$ to obtain

$$
a c x+b c y=e c \quad \text { and } \quad a c x+a d y=a f .
$$

Now subtract to obtain

$$
(b c-a d) y=e c-a f
$$

Back to the theory

Calculating $\Delta=\left|\begin{array}{rr}2 & 1 \\ 3 & -4\end{array}\right|=-11$. Since $\Delta \neq 0$ we can proceed with Cramer's solution.
$x=\frac{\left|\begin{array}{rr}7 & 1 \\ 5 & -4\end{array}\right|}{\left|\begin{array}{rr}2 & 1 \\ 3 & -4\end{array}\right|}, \quad y=\frac{\left|\begin{array}{ll}2 & 7 \\ 3 & 5\end{array}\right|}{\left|\begin{array}{rr}2 & 1 \\ 3 & -4\end{array}\right|}$
i.e. $x=\frac{(-28-5)}{(-8-3)}, \quad y=\frac{(10-21)}{(-8-3)}$ implying: $x=\frac{-33}{-11}=3, \quad y=\frac{-11}{-11}=1$.

Back to the theory

You should have checked $\left|\begin{array}{ll}2 & -3 \\ 4 & -6\end{array}\right|$ first, since
$\left|\begin{array}{ll}2 & -3 \\ 4 & -6\end{array}\right|=-12-(-12)=0$. Hence there is no unique solution in either case.
In the system (a) the second equation is twice the first so there are infinitely many solutions. (Here we can give $y$ any value we wish, $t$ say; but then the $x$ value is always $(6+3 t) / 2$. So for each value of $t$ there are values for $x$ and $y$ which simultaneously satisfy both equations. There are an infinite number of possible solutions).

In (b) the equations are inconsistent (since the first is $2 x-3 y=6$ and the second is $2 x-3 y=5$ which is not possible.) Hence there are no solutions.

Back to the theory

$$
\Delta=\left|\begin{array}{rrr}
1 & -2 & 1 \\
2 & 1 & -1 \\
3 & -1 & 2
\end{array}\right|
$$

Expanding along the top row,

$$
\begin{aligned}
\Delta & =1 \times\left|\begin{array}{rr}
1 & -1 \\
-1 & 2
\end{array}\right|-(-2) \times\left|\begin{array}{rr}
2 & -1 \\
3 & 2
\end{array}\right|+1 \times\left|\begin{array}{rr}
2 & 1 \\
3 & -1
\end{array}\right| \\
& =1 \times(2-1)+2 \times(4+3)+1 \times(-2-3) \\
& =1+14-5=10
\end{aligned}
$$

Back to the theory
$x_{1}=\left|\begin{array}{rrr}3 & -2 & 1 \\ 5 & 1 & -1 \\ 12 & -1 & 2\end{array}\right| \div \Delta$
Back to the theory

The numerator is found by expanding along the top row to be

$$
\begin{aligned}
3 & \times\left|\begin{array}{rr}
1 & -1 \\
-1 & 2
\end{array}\right|-(-2) \times\left|\begin{array}{rr}
5 & -1 \\
12 & 2
\end{array}\right|+1 \times\left|\begin{array}{rr}
5 & 1 \\
12 & -1
\end{array}\right| \\
& =3 \times 1+2 \times 22+1 \times(-17) \\
& =30
\end{aligned}
$$

Hence $x_{1}=\frac{1}{10} \times 30=3$
Back to the theory

$$
\begin{aligned}
x_{2} & =\frac{1}{10}\left|\begin{array}{rrr}
1 & 3 & 1 \\
2 & 5 & -1 \\
3 & 12 & 2
\end{array}\right| \\
& =\frac{1}{10}\left\{1 \times\left|\begin{array}{rr}
5 & -1 \\
12 & 2
\end{array}\right|-3 \times\left|\begin{array}{rr}
2 & -1 \\
3 & 2
\end{array}\right|+1 \times\left|\begin{array}{rr}
2 & 5 \\
3 & 12
\end{array}\right|\right\} \\
& =\frac{1}{10}\{22-3 \times 7+9\}=1 \\
x_{3} & =\frac{1}{10}\left\{1 \times\left|\begin{array}{rr}
1 & 5 \\
-1 & 12
\end{array}\right|-(-2) \times\left|\begin{array}{rr}
2 & 5 \\
3 & 12
\end{array}\right|+3 \times\left|\begin{array}{rr}
2 & 1 \\
3 & -1
\end{array}\right|\right\} \\
& =\frac{1}{10}\{17+2 \times 9+3 \times(-5)\}=2
\end{aligned}
$$

Back to the theory

1. (a) $x=\frac{1}{2}, y=0$
(b) $\Delta=0$, no solution
(c) $x=-\frac{1}{22}, y=-\frac{3}{11}$
2. (a) $x_{1}=\frac{8}{3}, x_{2}=-4, x_{3}=\frac{4}{3}$
(b) $x_{1}=\frac{1}{2}, x_{2}=\frac{1}{2}, x_{3}=1$

Back to the theory

